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STATIONARY OSCILLATIONS OF THERMOELASTIC ROD UNDER ACTION OF EXTERNAL DISTURBANCES

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ABSTRACT
The dynamics of a thermoelastic rod of finite length are studied under action of periodic external force and thermal sources. The model of connected thermoelasticity are used, taking into account the effect of temperature on the elastic deformation and stresses and as well as the effect of elastic deformation speed on the temperature field in the rod. Based on the method of generalized functions the analytical solutions of boundary value problems of the stationary vibration of a thermoelastic rod for various types of boundary conditions are constructed. The computer implementation of the boundary value problems are performed. The calculation results of movement and temperature of the rod at different frequencies, and a comparative analysis of solutions are presented.

INTRODUCTION
The core design is widely used in engineering as coupling and transmission units for the structural elements for different machines and mechanisms. During the operation they are subjected to variable mechanical and thermal influences that create a complex stress-strain state in designing elements, depending on their temperature, which affect at their reliability and durability. Therefore, the determination of thermoelastic stresses in rod structures with regard to their mechanical properties (especially the thermoelasticity) refers to actual scientific and technical problem.

Mathematical modeling of thermodynamic processes in rods leads to solving the boundary problems for thermoelastic media. There are various models of thermoelastic media. In the study of slow dynamic processes the unconnected thermoelasticity model are frequently used, which does not take into account the influence of elastic properties of medium on its temperature field. But fast vibratory processes in designs affect at the temperature field. In the study of such processes the coupled thermoelasticity model should be used, which is considered here to simulate the dynamics of thermoelastic rods.

In [1] there were constructed the fundamental and generalized solutions of thermoelasticity equations in a spatially one-dimensional case. They determine the thermoelastic stress-strain state of infinite thermoelastic rod under action of various periodic forces and heat sources. They may be described by distributions, both regular and singular, allowing to research the effect of concentrated sources of various types. Here we consider the boundary value problems (BVPs) of the dynamics of a thermoelastic rod of finite length at stationary oscillations. Four types of boundary conditions at each end of the rod have been considered. Based on the method of generalized functions the analytical solutions of BVPs are constructed and investegated.

THE MOTION EQUATIONS OF A THERMOELASTIC ROD
HARMONIC OSCILLATIONS
Let consider a thermoelastic rod with the length $2L$. The longitudinal displacements of a rod u and rod temperature θ are described by the mixed hyperbolic – parabolic equations of second order in the form [2]:
for $|x| \le L, t \ge 0$

$$\rho c^2 u_{,xx} - \rho u_{,tt} - \gamma \theta_{,x} + \rho F_1(x,t) = 0$$

$$\theta_{,xx} - \kappa^{-1}\theta_{,t} - \eta u_{,xt} + F_2(x,t) = 0$$

(1)

here $\theta(x,t)$ is relative temperature $\left(\theta = T(x,t) - T(x,0)\right)$, T is absolute temperature, ρ is the linear density, rigidity EJ and thermoelastic constants γ, η, κ are given, c is the speed of elastic

waves in a rod: $c = \sqrt{\dfrac{EJ}{\rho}}$. The symbol after the comma denotes the partial derivative with respect to the

specified index variable (for example $u,_x = \dfrac{\partial u}{\partial x}$, $u,_{xt} = \dfrac{\partial^2 u}{\partial x \partial t}$).

Let assume that the rod is subjected to the actions of periodic longitudinal force $F_1(x,t)$ and heat source $F_2(x,t)$ of the type:

$$F_1(x,t) = F_1(x)\exp(-i\omega t), \quad j = 1,2, \tag{2}$$

ω is the frequency of oscillation.

Thermoelastic stresses in the rod $\sigma(x,t)$ are defined by the Duhamel-Neumann low:

$$\sigma = \rho c^2 u,_x - \gamma \theta \tag{3}$$

The boundary conditions at the ends of the rod may be different. Here we formulate them for four boundary value problems (BVP) which are taken in the classical theory of thermoelasticity [1]:

1 BVP $\qquad u(x_j,t) = w_j \exp(-i\omega t), \quad \theta(x_j,t) = \theta_j \exp(-i\omega t); \quad j = 1,2 \tag{4$_1$}$

2 BVP $\qquad \sigma(x_j,t) = P_j \exp(-i\omega t), \quad \theta,_x(x_j,t) = q_j \exp(-i\omega t); \quad j = 1,2 \tag{4$_2$}$

3 BVP $\qquad u(x_j,t) = w_j \exp(-i\omega t), \quad \theta,_x(x_j,t) = q_j \exp(-i\omega t); \quad j = 1,2 \tag{4$_3$}$

4 BVP $\qquad \sigma(x_j,t) = P_j \exp(-i\omega t), \quad \theta(x_j,t) = \theta_j \exp(-i\omega t); \quad j = 1,2 \tag{4$_4$}$

where w_j, θ_j, P_j, q_j are the complex amplitudes of displacements, temperature, stresses and heat flow on the ends of the rod.

Also we can consider the boundary value problems with the one type of boundary conditions on the left end of the rod (any from (4)) and other type of boundary conditions on the right end.

By virtue of the harmony of acting forces (2) and boundary conditions (4_1-4_4), the solution can be presented in the same form:

$$(u,\theta) = (u(x), \theta(x))\exp(-i\omega t)$$

where the complex amplitudes satisfy the following system of differential equations:

$$\rho c^2 u,_{xx} + \rho \omega^2 u - \gamma \theta,_x + F_1(x) = 0,$$

$$\theta,_{xx} + i\omega \kappa^{-1}\theta + i\omega \eta u,_x + F_2(x) = 0. \tag{5}$$

We define the complex amplitudes of solutions satisfying (5) and one of the conditions (4), respectively to solved BVP.

METHOD OF GENERALIZED FUNCTIONS. ANALYTICAL SOLUTION

To solve the BVP we used the method of generalized functions [3,4]. On this basis, using a matrix of fundamental solutions $U(x,t)$, the analytical solution can be represented in the form

[4]: for $|x| \le L,\ t \ge 0$

$$u(x) = F_1 * U_1^1 + F_2 * U_1^2 +$$

$$+c^2 \sum_{k=1}^{2} (-1)^{k+1} \left\{ \left(p_k - \overset{\backprime}{\gamma} \theta_k \right) U_1^1(x - (-1)^k L, \omega) + u_k(\omega) U_1^1,_x (x - (-1)^k L, \omega) \right\} + \qquad (6)_1$$

$$+\sum_{k=1}^{2} (-1)^{k+1} \left\{ \left(q_k + i\omega\eta\, w_k \right) U_1^2\left(x - (-1)^k L, \omega \right) + \theta_k(\omega) U_1^2,_x \left(x - (-1)^k L, \omega \right) \right\}$$

$$\theta(x) = F_1 * U_2^1 + F_2 * U_2^2 +$$

$$+c^2 \sum_{k=1}^{2} (-1)^{k+1} \left\{ \left(p_k - \gamma\theta_k \right) U_2^1(x - (-1)^k L, \omega) + w_k U_2^1,_x (x - (-1)^k L, \omega) \right\} + \qquad (6)_2$$

$$+\sum_{k=1}^{2} (-1)^{k+1} \left(q_k + i\omega\eta w_k \right) U_2^2\left(x - (-1)^k L, \omega \right) + \theta_k U_2^2,_x \left(x - (-1)^k L, \omega \right)$$

$\overset{\backprime}{\gamma} = \gamma / \rho c^2$. Here the convolution for regular forces and heat sources are calculated by formulae:

$$F_j * U_k^j = H\left(L - \|x\| \right) \sum_{j=1}^{2} \int_{-L}^{L} F_j(y) U_k^j(x - y, \omega) dy \qquad (7)$$

$H(x)$ is Heaviside function. For singular $\hat{F}_1, \hat{F}_2$ you should use the definition of convolution (see [3]).

Formulas (6_1)- (6_2) determine the displacement and the temperature inside the rod when the displacements, stresses, temperatures and heat flows at the ends of a rod are known. However, for each boundary value problem it's known only four boundary values of the complex amplitudes. For remained four unknown boundary conditions we can get the resolving system of equations, based on the boundary conditions at the ends of the rod and the asymptotic properties of Green matrix $U(x,t)$ and its derivatives at $x = 0$.

THE MATRIX OF FUNDAMENTAL SOLUTIONS $U(x,\omega)$

The fundamental matrix $U(x,\omega)$ was constructed earlier in [1] by use Fourier transform of distributions. It has the following form:

$$U_1^j(x,\omega) = \frac{\delta_1^j \operatorname{sgn}(x)}{2(\lambda_1 - \lambda_2)} \left\{ i\omega\kappa^{-1} \left(\frac{\sin x\sqrt{\lambda_2}}{\sqrt{\lambda_2}} - \frac{\sin x\sqrt{\lambda_1}}{\sqrt{\lambda_1}} \right) + \left(\sqrt{\lambda_1} \sin x\sqrt{\lambda_1} - \sqrt{\lambda_2} \sin x\sqrt{\lambda_2} \right) \right\} -$$

$$-\frac{\gamma\delta_2^j \operatorname{sgn}(x)}{2(\lambda_1 - \lambda_2)} \left(\cos x\sqrt{\lambda_1} - \cos x\sqrt{\lambda_2} \right), \quad j = 1, 2$$

$$U_2^j(x,\omega) = \frac{\operatorname{sgn}(x)}{2(\lambda_1 - \lambda_2)} \left\{ i\omega\eta\delta_1^j \left(\cos x\sqrt{\lambda_1} - \cos x\sqrt{\lambda_2} \right) - \omega^2 \left(\frac{\sin x\sqrt{\lambda_1}}{\sqrt{\lambda_1}} - \frac{\sin x\sqrt{\lambda_2}}{\sqrt{\lambda_2}} \right) \delta_2^j + \right.$$

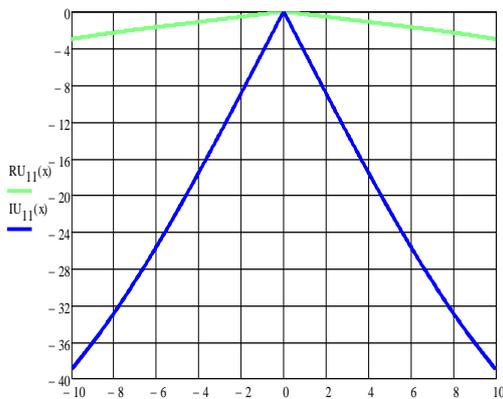$$\left. + c^2 \left(\sqrt{\lambda_1} \sin x\sqrt{\lambda_1} - \sqrt{\lambda_2} \sin x\sqrt{\lambda_2} \right) \delta_2^j \right\}, \quad j = 1, 2$$

where λ_1, λ_2 are the roots of characteristic equations of the system (5):
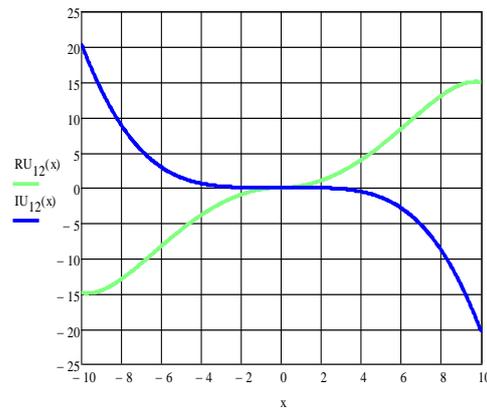
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$$\lambda_{1,2} = \frac{\omega}{2c^2}\left\{\left(\omega+i\gamma\eta\right)+ic^2k^{-1} \pm \sqrt{\left(\omega+i\left(\gamma\eta+c^2k^{-1}\right)\right)^2 - 4i\omega c^2 k^{-1}}\right\}$$
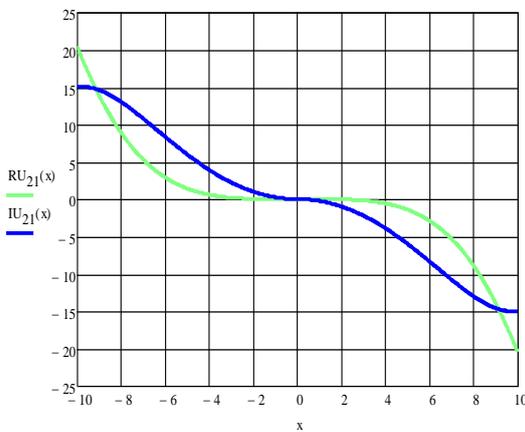
They depend only on the three thermoelastic constants: $c, \quad \alpha = \gamma\eta, \quad \beta = c^2 k^{-1}$. Dimension $[\alpha]=[\beta]=[\omega]$.

The components U_k^j are continuous at a point x = 0, and their derivatives at this point have a discontinuity of the first kind:
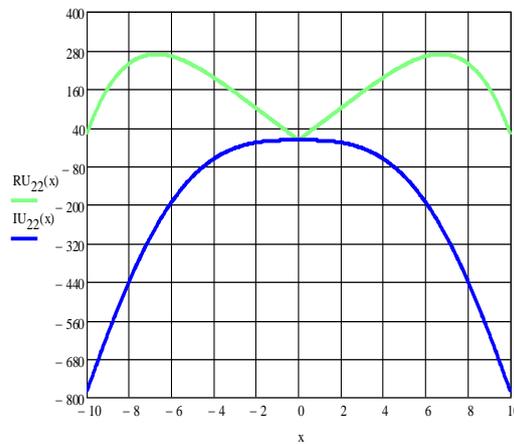

a

b

c

d

Figure 1 - Components U by ω =1   (γ=0.1,   c=1, k=1, η=1).

$$\overset{\%j}{U}_1,_x(\pm 0,\omega) = \pm\frac{1}{2}\delta_1^j, \; \overset{\%j}{U}_2,_x(\pm 0,\omega) = \pm\frac{c^2}{2}\delta_2^j \qquad (8)$$

Here

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$$\partial_x U_1^j(x,\omega) =$$

$$\left[\frac{(\lambda_1 - i\omega\kappa^{-1})}{2(\lambda_1 - \lambda_2)}\left(\cos x\sqrt{\lambda_1} - \cos x\sqrt{\lambda_2}\right) + \cos x\sqrt{\lambda_2}\right]\mathrm{sgn}(x)\delta_1^j -$$

$$+ \frac{\gamma}{2(\lambda_1 - \lambda_2)}\left(\sqrt{\lambda_1}\sin|x|\sqrt{\lambda_1} - \sqrt{\lambda_2}\sin|x|\sqrt{\lambda_2}\right)\delta_2^j$$

$$\partial_x U_2^j(x,\omega) =$$

$$= -\delta_1^j \frac{i\omega\eta\left(\sqrt{\lambda_1}\sin|x|\sqrt{\lambda_1} - \sqrt{\lambda_2}\sin|x|\sqrt{\lambda_2}\right)}{2(\lambda_1 - \lambda_2)} +$$

$$-\delta_2^j \mathrm{sgn}(x)\left\{\frac{\omega^2 - \lambda_1 c^2}{2(\lambda_1 - \lambda_2)}\left(\cos x\sqrt{\lambda_1} - \cos x\sqrt{\lambda_2}\right) - c^2 \cos x\sqrt{\lambda_2}\right\}$$

Their asymptotic properties are described in detail in [1].

At figures 1 you can see U ($\mathrm{RU}_{ij} = \mathrm{Re}(U_i^j)$, $\mathrm{IU}_{ij} = \mathrm{Im}(U_i^j)$), which describe the displacements and temperature of rod sections in point x in time moments

$$t_n = \frac{2\pi}{\omega}n, \quad t_k = \frac{2\pi}{\omega}k + \frac{\pi}{2\omega}, \quad n,k = 0,\pm1,\pm2...$$

Time t_n corresponds to real part U and t_k to its imaginary part.

THE RESOLVING EQUATIONS FOR BOUNDARY VALUE PROBLEMS

In the paper [4] it was shown, that the unknown amplitudes of the boundary functions of all BVPs satisfy to resolving system of equations, which can be represented in the matrix form:

$$\{A1\} \times \begin{Bmatrix} w_1 \\ p_1 \\ \theta_1 \\ q_1 \end{Bmatrix} + \{A2\} \begin{Bmatrix} w_2 \\ p_2 \\ \theta_2 \\ q_2 \end{Bmatrix} = b \tag{9}$$

where components of matrix A1 and A2 are equal to

$$A1 = \begin{Bmatrix} 0.5 & 0 & 0 & 0 \\ -\left(U_{1,x}^1 + i\omega\eta U_1^2\right)\Big|_{x=2L} & -U_1^1(2L,\omega) & \left(\gamma\bar{U}_1^1 - U_1^2,_x\right)\Big|_{x=2L} & -U_1^2(2L,\omega) \\ 0 & 0 & 0.5 & 0 \\ -\left(U_{2,x}^1 + i\omega\eta U_2^2\right)\Big|_{x=2L} & -U_2^1(2L,\omega) & \left(\gamma U_2^1 - U_2^2,_x\right)\Big|_{x=2L} & -U_2^2(2L,\omega) \end{Bmatrix}$$

$$A2 = \left\{ \begin{array}{cccc} \left(U_1^1,_x + i\omega\eta U_1^2\right)\Big|_{x=-2L} & U_1^1(-2L,\omega) & -\left(\gamma U_1^1 - U_1^2,_x\right)\Big|_{x=-2L} & U_1^2(-2L,\omega) \\ 0.5 & 0 & 0 & 0 \\ \left(U_2^1,_x + i\omega\eta U_2^2\right)\Big|_{x=-2L}, & U_2^1(-2L,\omega) & -\left(\gamma U_2^1 - U_2^2,_x\right)\Big|_{x=-2L} & U_2^2(-2L,\omega) \\ 0 & 0 & 0.5 & 0 \end{array} \right\}$$

Components of vector b depend on acting external forces and heat sources:

$$b = \left\{ \begin{array}{c} \left(F_1 * U_1^1 + F_2 * U_1^2\right)\Big|_{x=-L} \\ \left(F_1 * U_1^1 + F_2 * U_1^2\right)\Big|_{x=-L} \\ \left(F_1 * U_2^1 + F_2 * U_2^2\right)\Big|_{x=L} \\ \left(F_1 * U_2^1 + F_2 * U_2^2\right)\Big|_{x=L} \end{array} \right\}$$

Here we denote $(....)\Big|_{x=\pm L}$, the value of expressions in corresponding point $x = \pm L$.

This system from 4 linear algebraic equations connects 8 boundary values of complex amplitudes of displacements, stresses, temperature and heat flows.

It's easy to build resolving system of linear algebraic equations for any of the considered boundary value problems (4_1), (4_2), (4_3), (4_4) leaving in the left side of terms with unknown boundary values of the unknown functions and shifting to the right side with the known boundary values.

BOUNDARY VALUE PROBLEM 1 AND ITS SOLUTION
We consider here the first boundary value problem by known temperatures and the movement at the ends of the rod (4_1),. In this case, the resolving system of equations has the form:

$$\{A(L,\omega)\}_{4\times4} \times \begin{pmatrix} p_1 \\ q_1 \\ p_2 \\ q_2 \end{pmatrix} = \{B(L,\omega)\}_{4\times4} \begin{pmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} \qquad (10)$$

where the matrices A and B are expressed through the elements of the matrices A1 and A2 so

$$A = \left\{ \begin{array}{cccc} A1_{12} & A1_{14} & A2_{12} & A2_{14} \\ A1_{22} & A1_{24} & A2_{22} & A2_{24} \\ A1_{32} & A1_{34} & A2_{32} & A2_{34} \\ A1_{42} & A1_{44} & A2_{42} & A2_{44} \end{array} \right\} \qquad B = -\left\{ \begin{array}{cccc} A1_{11} & A1_{13} & A2_{11} & A2_{13} \\ A1_{21} & A1_{23} & A2_{21} & A2_{23} \\ A1_{31} & A1_{33} & A2_{31} & A2_{33} \\ A1_{41} & A1_{43} & A2_{41} & A2_{43} \end{array} \right\}$$

Solving this system we determine unknown stresses and heat flow on the ends of a rod. Then, using (6_1), (6_2), we calculate displacement and temperature in any point of a rod.

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The calculations were performed for a medium with dimensionless parameters: $L = 1, \gamma = 1, \eta = 1, k = 1, c = 3, \rho = 1$.

On the figures 3, 5, 7,9 (a, b) you see the real and imaginary parts of the complex amplitudes of displacement and temperature, which describe the real state of the rod at $t = 0 + 2\pi n/\omega$ (a quarter of the period).On the figures 4,6,8,10 (a,b) there are the amplitudes of the displacements and the temperature along the rod by different frequencies: w = 0.1,1,10,100.

Here you can see the formation of *standing waves*. At low frequencies $\left(\omega = 0.1\right)$ the middle of the rod is fixed, the maximum longitudinal displacement observed on the last quarter of the length of the rod. A max temperature observed   in the middle of the rod. At low frequencies, the maximum temperature in the middle of the rod is higher than the temperature at its ends.
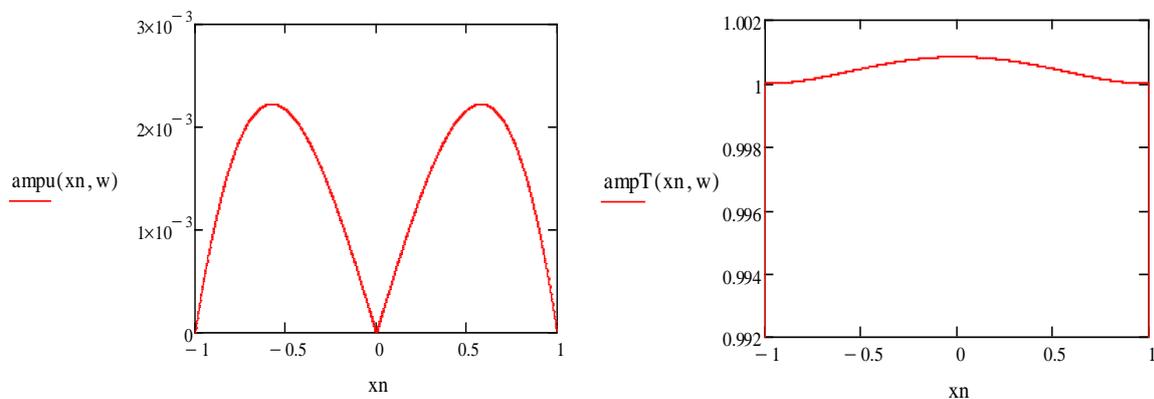
                    a   *b*
***Figure 2 - The amplitudes of the displacement (a) and the temperature (b)
along the rod: ω =0.1***

By increasing the frequency the number of local extremum increases, the amplitude of the temperature increase in comparison with its value at the ends of the rod. There is a nodal point where the displacement and temperature are close or equal to zero. But extremes of amplitudes and temperatures are shifted relative to each other (where displacements are zero there are a maximum of amplitudes of the temperature).

In the table 1 it's shown the maximum amplitude of the displacement and the temperature in the considered frequency range. With increasing the frequency the amplitude of the movement rises sharply and then begins to fall. The same is observed for temperature. By temperature fluctuations at the ends the maximum amplitude of the temperature fluctuations in the rod increased about 20 percentages.
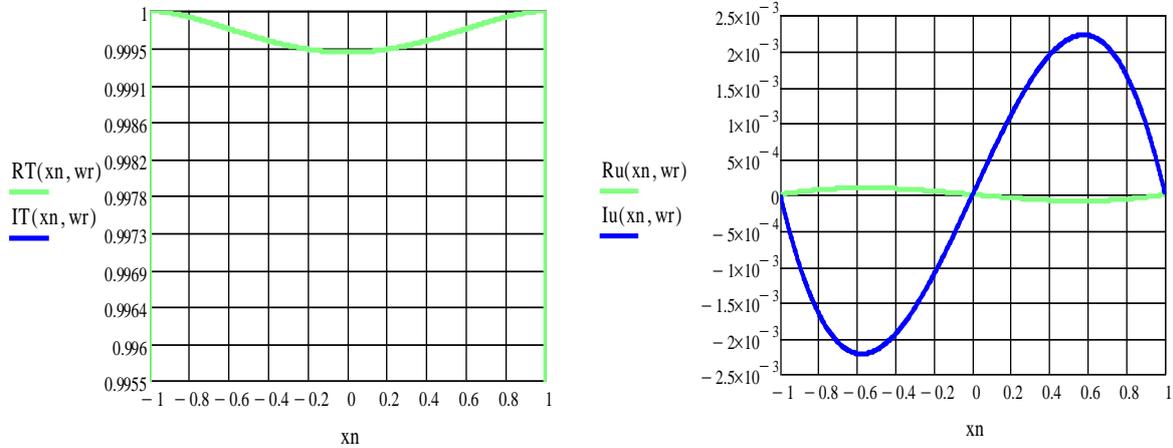
Figure 3 - Displacements and temperature of the rod at t=2πn/ω and t=2πn/ω+π/2ω: $\omega = 0.1$

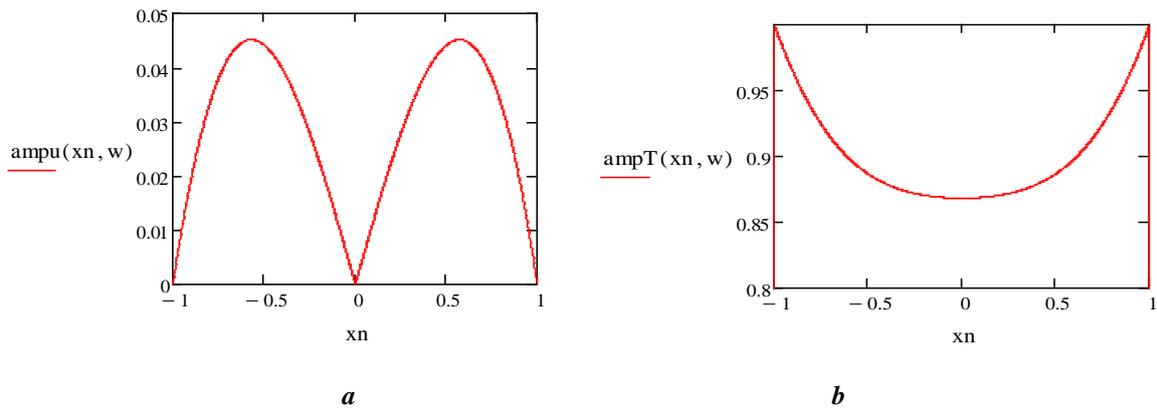*a*   *b*
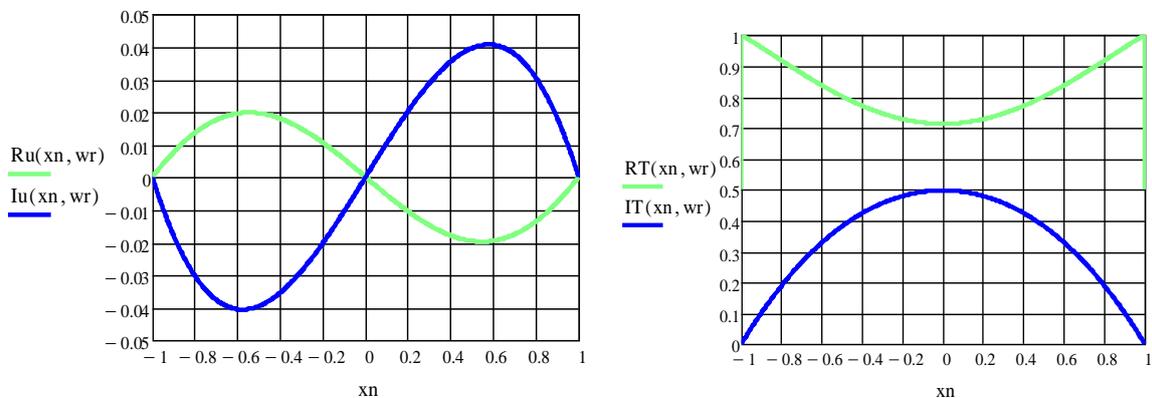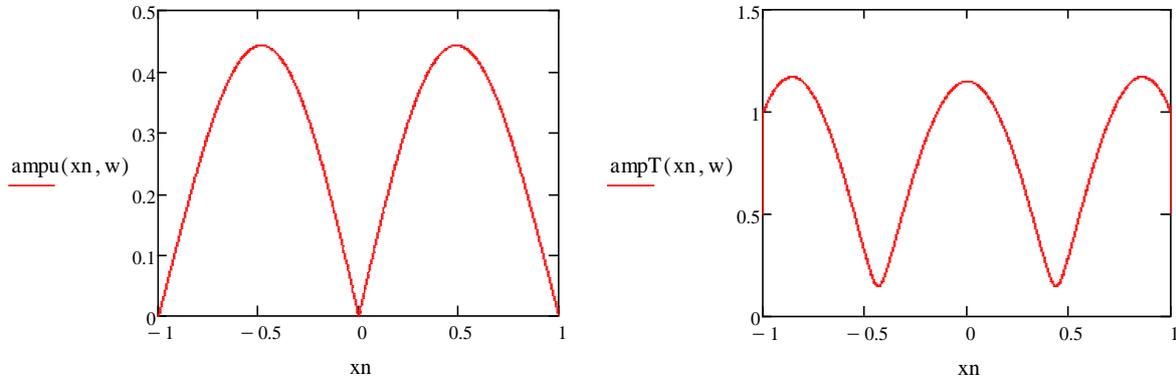*Figure 4 - The amplitudes of displacement (a) and temperature (b) along the rod:* $\omega = 1$

Figure 5 - Displacements and temperature of the rod at t=2πn/ω and t=2πn/ω+π/2ω: $\omega = 1$

Figure 6 - The amplitudes of displacement (a) and temperature (b) of the rod: ω =10

a   *b*
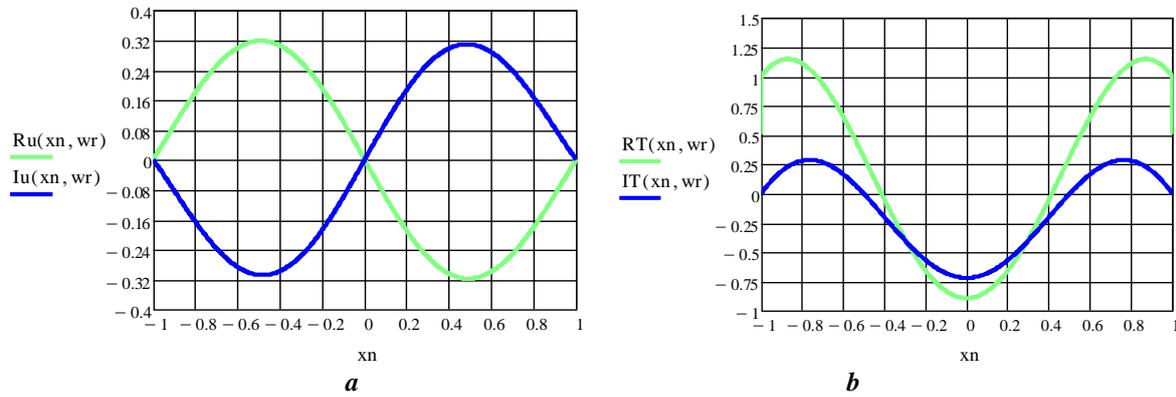
Figure 7 - Displacements and temperature of the rod at t=2πn/ω and t=2πn/ω+π/2ω: $\omega = 10$
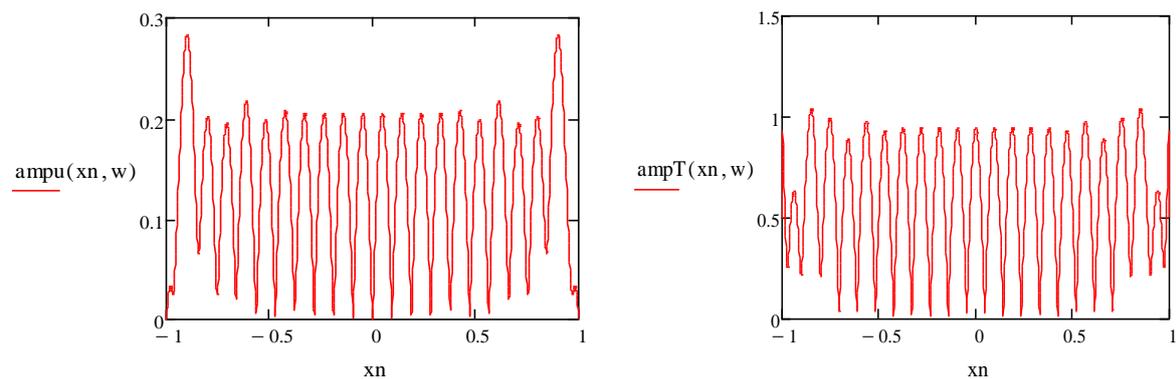

Figure 8 - The amplitudes of displacement (a) and temperature (b) of the rod: ω =100

a   *b*
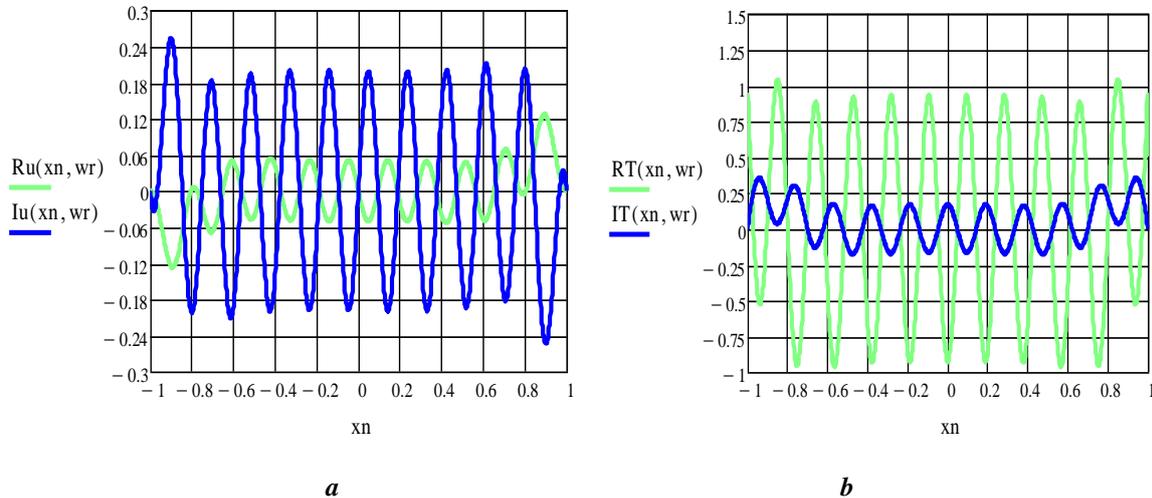
Figure 9 - Displacements and temperature of the rod at t=2πn/ω and t=2πn/ω+π/2ω: $\omega = 100$

Table 1. The maximum amplitude of the displacements and temperature

ω	U max	T max
0.1	0.0022	1.001
1	0.032	1.168
10	0.443	1.2
100	0.28	1.04

PERIODIC OSCILLATION OF THERMOELASTIC RODE

At more difficult periodic processes it is necessary to write the boundary conditions and the operating sources into Fourier's series on time. Let τ is period of oscillation of external actions:

$$F_j(x,t) = F_j(x, t + \tau n), \quad n = 0, \pm 1, \pm 2, \pm 3, ..., \quad j = 1, 2$$

and given boundary conditions have the same period:

$$f_j(x,t) = f_j(x, t + \tau n), \quad n = 0, \pm 1, \pm 2, \pm 3, ..., \quad j = 1, 2, 3, 4$$

Then they can be presented in the form of Fourier series:

$$F_j(x_l, t) = \sum_n F_{jn}(x) \exp(-i\omega_n t), \quad j = 1, 2,$$

$$f_j(x_l, t) = \sum_n f_{jn} \exp(-i\omega_n t), \quad j = 1, 2, 3, 4, \quad l = 1, 2$$

where

$$F_{jn}(x) = \tau^{-1} \int_0^\tau F_j(x,t) \exp(i\omega_n t) dt, \quad j=1,2,$$

$$f_{jn}(x_l) = \tau^{-1} \int_0^\tau f_j(x_l,t) \exp(i\omega_n t) dt, \quad j=1,2,3,4, \quad l=1,2$$

Here $\omega_0 = 2\pi\tau^{-1}$ is basic frequency of oscillation, $\omega_n = n\omega_0 = 2\pi n / \tau$.

Then we can build the decision of BVP in the form of Fourier's series

$$u(x,t) = \sum_n u_n(x) \exp(-i\omega_n t), \quad \theta(x,t) = \sum_n \theta_n(x) \exp(-i\omega_n t)$$

and to find complex amplitude functions $u_n(x), \theta_n(x)$ for each harmonica, following to this method.

CONCLUSION

Using the system of equations (9) it is possible to build solutions of any of the set BVPs 1-4, and also with the mixed conditions on the ends of a rod. Similarly on a basis this system of equations the semi-inverse and inverse BVPs can be solved, when number of boundary conditions on the different ends of a rod is various. The main thing - them has to be four.

The offered calculation procedure has to find application at research of support of buildings and constructions, thermos tension state of rods designs of different functions under actions of external forces of different nature and also by their heating and cooling.

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